KSME Journal, Vol. 2, No. 2, pp. 133~139, 1988.

THE EFFECTS OF THERMOELASTIC AND WEAR ON THE LEAKAGE OF COMPRESSIBLE GASES IN SHAFT SEALS

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(Received October 14, 1988)

Analysis are presented for a compressible gas flow across shaft seals. The leakage flow rate is developed analytically for a sealing gap due to eccentricity, misalignment of a shaft, sinusoidal wavy surfaces for both bodies, thermoelastic deformation on the edge of the body, and wear on the mating surfaces. A pressure distribution is determined as a power series based on the simplified nonlinear Reynolds equation. The restriction to exclude the thermoelastic effects was presented. Analytical results indicate that the thermoelastic distortion may dominate the seal performance when the eccentricity is large and width of the seal small at high speeds.

Key Words: Shaft Seal, Sealing Gap, Thermoelastic Effects, Leakage Flow Rate, Surface Waveness.

NOMENCLATURE

e_0 : Shaft eccentricity at $z=0$
$\bar{h}$: r_1-r_2, Mean sealing gap at $z=0$
$\tilde{h}$: Sealing gap variation due to wavy surfaces
$\tilde{h}_t$: Sealing gap variation due to the eccentricity and tilted shaft
m : Exponent determined by the fluid property
n : Number of waves
r : Radius
R_g : Gas constant
t : Time
T : Temperature
w : Wear coefficient
β : T/T_l
β_λ : T_u/T_l
γ : Angle of tilt
ε : $\gamma L/\bar{h}$, Tilt parameter of the shaft seal
ε_0 : $e_0/\bar{h}$, Eccentricity ratio at $z=0$
χ : $2\pi/\lambda$, Wave number
λ : Wavelength
Λ_l : T_l/T_r
Λ_u : T_u/T_r
ξ_l : $|\tilde{h}_l|\bar{h}$
ξ_u : $|\tilde{h}_u|/\bar{h}$
ϕ : Angle between the line of centers
$\tilde{\Phi}_l$: $\Omega_l t+\Phi_l$
$\tilde{\Phi}_u$: $\Omega_u t-\Phi_u$
Ψ : Attitude angle
Ω_l : $\chi_l c_l$
Ω_u : $\chi_u(c_u+U)$

Subscripts

l : Lower body
r : Reference conditions
u : Upper body

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1. INTRODUCTION

A primary objective of the seal design effort is to minimize the seal leakage flow rate and the power loss due to viscous friction. In order to achieve these conditions, it is necessary to combine actual factors that affect to a seal operation. For the purpose of analysis, the influence of operating conditions such as shaft speed and material removal on a mating surface as well as geometry effects done by Kim(1988) are included.

The geometry of misaligned shaft has been studied by Sassenfeld and Walther(1954). The wavy model of face seals done by Bryant and Kim(1987) was employed for this analysis. Work involving thermoelastic effects has been perofrmed by Burton et al.(Burton, Kilaparti, and Nerlikar, 1973 ; Burton and Nerlikar, 1974 ; Burton, 1975 ; Burton and Nerlikar, 1975). They analyzed the curvature produced on the edge of a tube due to a frictional heating.

The objective of the paper is accurately to estimate the leakage flow rate for compressible fluid flow across shaft seals. The mean film thickness, eccentricity and misalignment of the shaft, wavy surfaces, thermoelastic, and wear deformations on the edge of the body are included for a practical seal analysis.

2. ANALYSIS OF SEALING GAP

To analyze a seal performance more accurately, it will be necessary to combine all the possible components such as the geometry and operating ralated components. The overall sealing gap, h around the sealing circumference may then be given by

$$h(\theta, z, t)=\bar{h}+\tilde{h}_t+\tilde{h}+\delta_{th}+\delta_w \tag{1}$$

where the first three terms representing geometric components are described in the previous study(Kim, 1988). Here δ_{th}

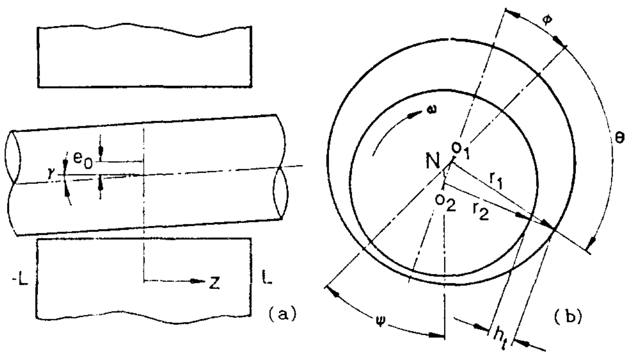

Fig. 1 Geometry of a misaligned shaft seal

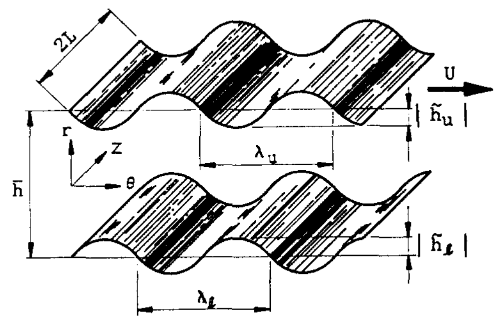

Fig. 2 Surface waveness of the shaft seal

is a thermoelastic displacement due to a viscous dissipation, and δ_w denotes displacements caused by a metal-to-metal contact between a pair of mating surfaces. Figs. 1 and 2 show the geometrical configurations of shaft seals.

2.1 Thermoelastic Displacement by Viscous Friction

When the heat generated by viscous friction across the film passes from the sealing gap to the shaft and seal, a thermal deformation on the edge of the body is due to uneven distribution of temperature on the boundary of the shaft seal. The seal is held stationary, while the shaft is made to rotate at an angular velocity, ω. The pressure gradient in the tangential direction is assumed to be much smaller than that of the axial direction for the small eccentricity of the shaft.

The sealing ratio, h/r is assumed to be very small enough for the film to be laminar viscous flow. And for $h \ll 2L$, the viscous heat, q across the sealing gap can be expressed as

$$q = \int_{r_u}^{r_l} \eta \left(\frac{\partial v_\theta}{\partial r} \right)^2 dr \tag{2}$$

where v_θ is the tangential velocity and η the viscosity of a fluid. A temperature dependent viscosity for compressible gases may be expressed as(Constantinescu, 1969)

$$\eta = \eta_r (T/T_r)^m. \tag{3}$$

The velocity gradient in the tangential direction derived by Eq. (16b) of the previous study(Kim, 1988) is reduced to

$$\frac{\partial v_\theta}{\partial r} = \frac{U(1-\beta^{m+1})}{(m+1)(\beta_\Lambda - 1)h} \beta^{-m}. \tag{4}$$

Substituting Eqs. (3) and (4) into Eq. (2) gives

$$q \cong \eta_r G_0 U^2 \left(\frac{1}{\bar{h}} - \frac{\tilde{h}_t + \tilde{h}}{\bar{h}^2} \right) \tag{5}$$

where

$$G_0 = \frac{\Lambda_l^m (1-\beta_\Lambda^{m+1})}{(m+1)(1-\beta_\Lambda)}. \tag{6}$$

In Eq. (5) the first term represents an uniform heating of the parallel surfaces. The second term is a non-uniform heating caused by the tilting and surface waveness. The uniform heating term should be eliminated from the overall viscous heat because it does not change the surface curvatures. Then non-uniform heating, $\hat{q}$ may be written as

$$\hat{q} = -\frac{\eta_r G_0 U^2}{\bar{h}} [\varepsilon_0 \cos\theta - \varepsilon(2Z-1)\cos(\theta - \Psi) + \tilde{h}]. \tag{7}$$

The thermoelastic distortion, δ_{th} produced on the edge of the body is given by (Burton, Kilaparti, and Nerlikar, 1973)

$$\frac{d^2 \delta_{th}}{d\theta^2} = \frac{\alpha r^2 \hat{q}}{K} \tag{8}$$

Here α is the coefficient of thermal expansion and K the thermal conductivity of the body. By integrating Eq. (8) with respect to θ, the thermoelastic displacement due to a thermal expansion on the boundary of the surface is found to be

$$\delta_{th} = -\frac{\eta_r G_0 r^2 U^2}{\bar{h}} (\delta_{tht} + \delta_{thu} + \delta_{thl}) \tag{9}$$

where

$$\delta_{tht} = \frac{\alpha_u}{K_u} \{\varepsilon_0 (1-\cos\theta) - \varepsilon(2Z-1)[1-\cos(\theta-\Psi)]\} \tag{10a}$$

$$\delta_{thu} = \frac{\alpha_u \xi_u}{K_u n_u} \left[\frac{\sin(n_u\theta - \Omega_u t + \phi)}{n_u} - \theta \right] \tag{10b}$$

$$\delta_{thl} = \frac{\alpha_l \xi_l}{K_l n_l} \left[\frac{\sin(n_l\theta - \Omega_l t + \phi)}{n_l} - \theta \right] \tag{10c}$$

2.2 Wear Displacement

When the wearing of the shaft seal is considered, the wear displacements due to the material removals on the edge of the mating surfaces is related to the metal-to-metal contact pressure. This pressure on the contacting spots may be obtained from the temperature distribution produced solely by frictional dissipation on the surfaces(Burton and Nerlikar, 1974 ; Burton and Heckmann, 1975). The heat flow in the radial direction is given by

$$q = -K \frac{\partial T}{\partial r} = \mu_a p_a U \tag{11}$$

where μ_a is the coefficient of friction. The temperature gradient in the radial direction will be given by differentiating Eqs. (49) and (50) with respect to r for the upper and lower surfaces, respectively (Kim and Burton, 1986). The contact pressure can then be expressed as

$$P_a = \frac{1}{\mu_a U} [|q_u| \sin(n_u\theta + \phi - \tilde{\Phi}_u) + |q_l| \sin(n_l\theta + \phi - \tilde{\Phi}_l)] \tag{12}$$

where

$$|q_u| = K_u |T_u| (a_u^2 + b_u^2)^{1/2} \quad (13a)$$

$$\Phi_u = \tan^{-1}\left(-\frac{a_u}{b_u}\right) \quad (13b)$$

$$a_u = \left\{-\frac{\chi_u}{2} + \frac{\chi_u}{2}\left[\chi_u^2 + \left(\frac{c_u + U}{\alpha_u}\right)^2\right]^{1/2}\right\}^{1/2} \quad (13c)$$

$$b_u = \left\{\frac{\chi_u}{2} + \frac{\chi_u}{2}\left[\chi_u^2 + \left(\frac{c_u + U}{\alpha_u}\right)^2\right]^{1/2}\right\}^{1/2} \quad (13d)$$

$$|q_l| = K_l |T_l| (a_l^2 + b_l^2)^{1/2} \quad (13e)$$

$$\Phi_l = \tan^{-1}\left(-\frac{a_l}{b_l}\right) \quad (13f)$$

$$a_l = \left\{-\frac{\chi_l}{2} + \frac{\chi_l}{2}\left[\chi_l^2 + \left(\frac{c_l}{\alpha_l}\right)^2\right]^{1/2}\right\}^{1/2} \quad (13g)$$

$$b_l = \left\{\frac{\chi_l}{2} + \frac{\chi_l}{2}\left[\chi_l^2 + \left(\frac{c_l}{\alpha_l}\right)^2\right]^{1/2}\right\}^{1/2} \quad (13h)$$

The wave velocities, c_l and c_u within the body are given by Kim and Burton(1986). Eq. (12) will have an alternating sign depending on a sine function. Since the negative pressure can only cut out a material on the metallic contact area, the positive portion will be excluded from the asperity contact equation for the wear displacement, i.e., $P_a = 0$ will replace $P_a > 0$. This defines a new asperity pressure, P_a that is periodic in θ ; P_a may be expressed using the Fourier series

$$P_a = \frac{|q_u|}{\mu_a U}\left\{-\frac{1}{\pi} + \frac{1}{2}\sin(n_u\theta + \phi - \tilde{\Phi}_u) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos[2n(n_u\theta + \phi - \tilde{\Phi}_u)]}{4n^2 - 1}\right\} + \frac{|q_l|}{\mu_a U}\left\{-\frac{1}{\pi} + \frac{1}{2}\sin(n_l\theta + \phi - \tilde{\Phi}_l) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos[2n(n_l\theta + \phi - \tilde{\Phi}_l)]}{4n^2 - 1}\right\} \quad (14)$$

The contact pressure causes the wear displacement. Burwell and Strang(1952) addressed a simple theory for the calculation of wear displacement as

$$\delta_w = -w\int_0^t P_a U dt \quad (15)$$

Substituting the asperity pressure of Eq. (14) into Eq. (15) and integrating yields

$$\delta_w = \delta_{wu} + \delta_{wl} \quad (16)$$

where

$$\delta_{wu} = -\frac{w|q_u|}{\mu_a}\left\{-\frac{t}{\pi} - \frac{\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u} \times \sin\left(n_u\theta - \frac{\Omega_u t}{2} + \phi - \Phi_u\right) - \frac{2}{\pi\Omega_u}\sum_{n=1}^{\infty}\frac{\sin(n\Omega_u t)\cos\left[2n\left(n_u\theta - \frac{\Omega_u t}{2} + \phi + \Phi_u\right)\right]}{n[4n^2 - 1]}\right\} \quad (17a)$$

$$\delta_{wl} = -\frac{w|q_l|}{\mu_a}\left\{-\frac{t}{\pi} - \frac{\sin\left(\frac{\Omega_l t}{2}\right)}{\Omega_l} \times \sin\left(n_l\theta - \frac{\Omega_l t}{2} + \phi - \Phi_l\right) - \frac{2}{\pi\Omega_l}\sum_{n=1}^{\infty}\frac{\sin(n\Omega_l t)\cos\left[2n\left(n_l\theta - \frac{\Omega_l t}{2} + \phi - \Phi_l\right)\right]}{n(4n^2 - 1)}\right\} \quad (17b)$$

3. ANALYSIS OF SHAFT SEALS ACROSS THE SEALING GAP

The equations such as temperature, velocity, and pressure expressions developed in the previous study(Kim, 1988) are here applied to the overall sealing gap given Eq. (1).

3.1 Pressure Distribution

The pressure distribution function is determined by solving a normalized nonlinear Reynolds equation (21) in the previous paper(Kim, 1988) using the power series as

$$P = P_0 + P_1 Z + P_2 Z^2 + \cdots \quad (18)$$

where $0 \leq Z \leq 1$. The coefficients of Eq. (18) are determined with the same boundary conditions (28) as done in the previous study(Kim, 1988). Therefore we can use the recursive formula given by Eq. (26) in the previous study(Kim, 1988) for the unknown coefficients, P_j. Here

$$\bar{H}_0 = 1 + a_1|\bar{\varepsilon}|\cos(\theta - \bar{\Psi}) + a_2(\delta_{thu} + \delta_{thl}) + \frac{1}{\bar{h}}(\tilde{h} + \delta_w) \quad (19a)$$

$$\bar{H}_1 = -2\varepsilon\left[\frac{a_2\alpha_u}{K_u} + a_1\cos(\theta - \Psi)\right],\quad \bar{H}_2 = 0,\quad \bar{H}_3 = 0,\ \cdots \quad (19b)$$

$$A_0 = \frac{\xi_u n_u}{\Lambda_u \Gamma_1}\cos(n_u\theta - \Omega_u t + \phi) + \frac{\Gamma_2}{\Gamma_1}\{a_1|\bar{\varepsilon}|\sin(\theta - \bar{\Psi}) - a_2[n_u|\delta_{thu}|\cos(n_u\theta - \Omega_u t + \phi) + n_l|\delta_{thl}|\cos(n_l\theta - \Omega_l t + \phi) - a_3]\} - \frac{1}{\bar{h}}\frac{d}{d\theta}(\tilde{h} + \delta_w) \quad (20a)$$

$$A_1 = -\frac{2a_1\varepsilon\Gamma_2}{\Gamma_1}\sin(\theta - \Psi),\quad A_2 = 0,\quad A_3 = 0,\ \cdots \quad (20b)$$

where Γ_1 and Γ_2 are given in the previous study(Kim, 1988).

$$a_1 = 1 - \frac{a_2\alpha_u}{K_u},\quad a_2 = -\eta_r G_0\left(\frac{rU}{\bar{h}}\right)^2,\quad a_3 = \frac{\alpha_u\xi_u}{K_u n_u} + \frac{\alpha_l\xi_l}{K_l n_l}$$

$$|\bar{\varepsilon}| = (\varepsilon_0^2 + \varepsilon^2 + 2\varepsilon_0\varepsilon\cos\Psi)^{1/2},$$

$$\bar{\Psi} = \tan^{-1}\left(\frac{\varepsilon\sin\Psi}{\varepsilon_0 + \varepsilon\cos\Psi}\right)$$

$$|\delta_{thu}| = \frac{\alpha_u\xi_u}{K_u n_u^2},\quad |\delta_{thl}| = \frac{\alpha_l\xi_l}{K_l n_l^2}$$

3.2 Leakage Flow Rate

The mass leakage rate, $\dot{m}$ in the axial direction is given by

$$\dot{m} = n\int_0^{\frac{2\pi}{n}}\int_{r_l}^{r_u}\rho r v_z dr d\theta \quad (21)$$

The dimensionless form of the above equation is given by the equation (32) in the previous study(Kim, 1988). The equation may be simplified with $Z = 0$ because it can not vary with Z. Thus the dimensionalized form of Eq. (21) is simplified as

$$\frac{R_g T_r r_l \bar{h}}{8\eta_r U^2 L^3}\dot{m} = \dot{M} = \Gamma_1 n\int_0^{\frac{2\pi}{n}}\bar{H}_0^3 P_0 P_1 d\theta. \quad (22)$$

Integrating Eq. (22) with respect to θ yields

$$\dot{M}=\Gamma_1 P_0 P_1 (I_{a3}+I_{c3}+I_{d3}+I_{e3}+I_{3a}+I_{3b}+I_{3c} + I_{3d}+I_{3e}+I_{6a}+I_{6b}+I_{6c}+I_{6d}) \tag{23}$$

where I_{ab} is an integration functions given in the Appendix. $J_{11}\sim J_{13}$, $J_{36}\sim J_{38}$, and $J_{9l}\sim J_{9u}$ are given in the previous study(Kim, 1988).

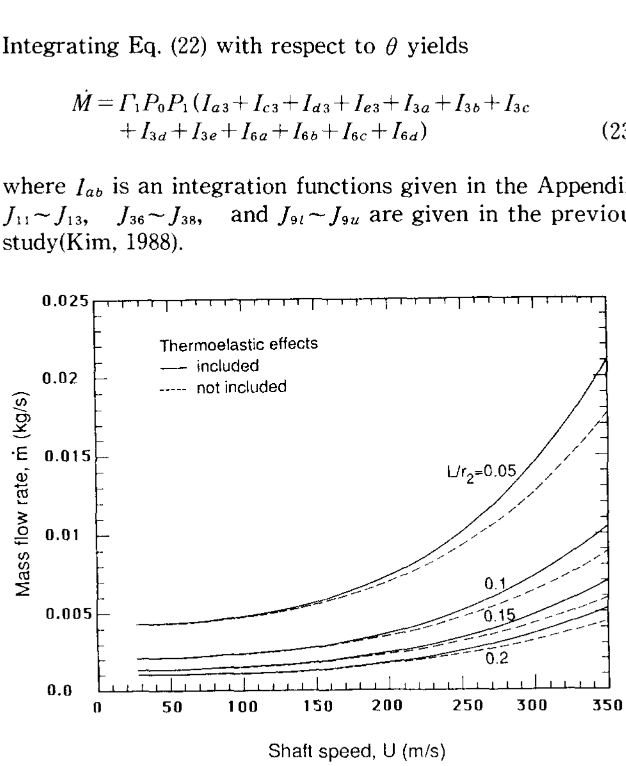

Fig. 3 Mass flow rate as a function of speed for various L/r_2 ratios and $\varepsilon_0=0.38$

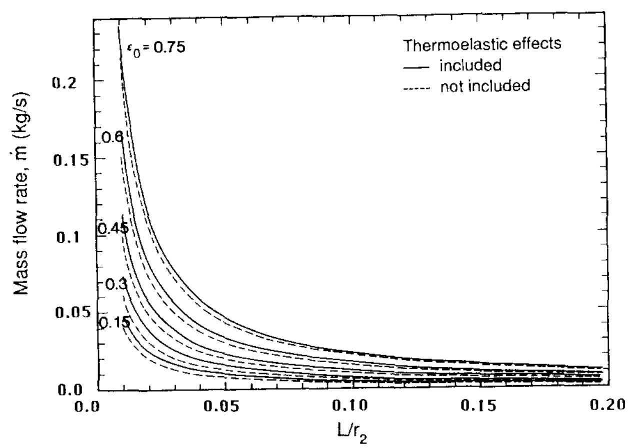

Fig. 4 Mass flow rate as a function of L/r_2 ratios for various values of eccentricity ; $U=333$m/s, $\bar{h}=50\mu$m

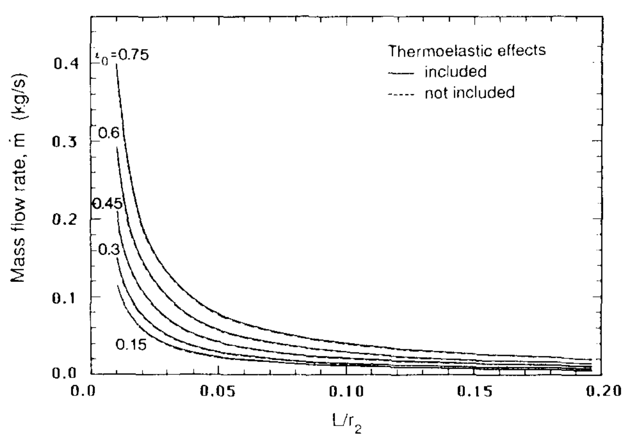

Fig. 5 Mass flow rate as a function of L/r_2 ratios for various values of eccentricity ; $U=333$m/s, $\bar{h}=70\mu$m

4. RESULTS AND DISCUSSION

The data used for the previous study(Kim, 1988) was selected for this study. This may explain the effects of geometry and running conditions. All the results presented in this paper were obtained for $\bar{h}=40\mu$m, $U=333$m/s, wear coefficient $w=5.5\times10^{-10}$m^2/N, and $\varepsilon_0=0.38$ unless otherwise stated.

Figure 3 shows a substantial increase in leakage flow rate for the small width of the seal, and clearly indicates the role of thermoelastic effects at high speeds, i.e., $U>150$m/s. According to the calculated results, the wear effect was a very small portion in comparsion to the thermoelastic distortion.

In Figs. 4 and 5, the mass flow rate hyperbolically decreases as the ratio of L/r_2 increases for $\bar{h}=50\mu$m and 70μm, respectively. As shown in Figs. 4 and 5, the high value of mean sealing gap appears to produce the remarkable increase in leakage rate. The thermoelastic effect is diminishing for the increased width of the shaft seal to the radius. Figures 4 and 5 show decreases in mass flow rate with reductions in eccentricity ratio, ε_0. The thermal term is not significient for the increased mean sealing gap shown in Fig. 5. As the eccentricity ratios as shown in Figs. 4 and 5 increases, the leakage flow rate may be dominated by the thermoelastic effects. Therefore, the restriction of the eccentricity and seal width may thus lead to an appreciable reduction of leakage rate.

5. CONCLUSIONS

An analytical method to estimate the leakage rate has been presented for compressible fluid flow across shaft seals with various sealing components such as the eccentricity, misalignment, surface waviness, thermoelastic, and wear deformations on the edge of the body.

According to the calculated results, the main factors which affect the seal performance at high speeds appear to be the width of the seal and eccentricity of the shaft. The thermoelastic effects on the mass flow rate are particularly increased for high eccentricity ratio and high speeds. Therefore to eliminate the thermoelastic effects, it is necessary to restrict the eccentricity ratio and increase the width of the seal.

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APPENDIX

Integration Functions for Mass Flow Rate

$$I_{a3}=2\pi$$

$$I_{c3}=a_2^3[-4\pi^4a_3^3+3|\delta_{thu}|^2(|\delta_{thl}|J_{11}-a_3J_{2u})$$
$$+3|\delta_{thl}|^2(|\delta_{thu}|J_{12}-a_3J_{2l})+3a_3^2(|\delta_{thu}|J_{1u}+|\delta_{thl}|J_{1l})$$
$$-6_{a3}|\delta_{thu}||\delta_{thl}|J_{3c}]$$

$$I_{d3}=-3\xi_u\xi_l(\xi_uJ_{11}+\xi_lJ_{12})$$

$$I_{e3}=\frac{1}{h^3}\Bigg\{\frac{2}{\pi^2}[t(|\delta_{wu}|+|\delta_{wl}|)]^3+3t(|\delta_{wu}|+|\delta_{wl}|)$$
$$\cdot\Bigg[\left(\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}\right)^2+\left(\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}\right)^2\Bigg]$$
$$+\frac{3|\delta_{wu}||\delta_{wl}|}{\Omega_u\Omega_l}\Bigg[|\delta_{wu}|\sin\left(\frac{\Omega_lt}{2}\right)\Bigg(\frac{2t\sin\left(\frac{\Omega_ut}{2}\right)}{\pi}J_{21}$$
$$-\frac{\sin^2\left(\frac{\Omega_ut}{2}\right)}{\Omega_u^2}J_{22}\Bigg]+|\delta_{wl}|\sin\left(\frac{\Omega_ut}{2}\right)\frac{2t\sin\left(\frac{\Omega_lt}{2}\right)}{\pi}J_{21}$$
$$-\frac{\sin^2\left(\frac{\Omega_lt}{2}\right)}{\Omega_l^2}J_{23}\Bigg\}\Bigg]\Bigg)$$

$$I_{3a}=6[-\pi^2a_3+t(|\delta_{wu}|+|\delta_{wl}|)]$$

$$I_{3b}=3b_1^2\Bigg\{\pi+a_2(|\delta_{thu}|J_{31}+|\delta_{thl}|J_{32}-a_3J_{33})-(\xi_uJ_{31}+\xi_lJ_{32})$$
$$+\frac{1}{h}\Bigg[t(|\delta_{wu}|+|\delta_{wl}|)-\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}J_{34}$$
$$-\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}J_{35}\Bigg]\Bigg\}$$

$$I_{3c}=3a_2^2\Bigg\{\pi\left(|\delta_{thu}|^2+|\delta_{thl}|^2+\frac{8\pi^2a_3^2}{3}\right)$$
$$+(|\delta_{thu}||\delta_{thl}|J_{13}-a_3|\delta_{thu}|J_{3u}$$
$$-a_3|\delta_{thl}|J_{3l}+b_1[|\delta_{thu}|^2J_{36}+|\delta_{thl}|^2J_{37}$$
$$+a_3^2J_{3b}+2(|\delta_{thu}||\delta_{thl}|J_{38}-a_3|\delta_{thu}|J_{4u}-a_3|\delta_{thl}|J_{4l})]$$
$$-\xi_u[|\delta_{thl}|^2J_{12}+a_3^2J_{1u}+2(|\delta_{thu}||\delta_{thl}|J_{11}-a_3|\delta_{thu}|J_{2u}$$
$$-a_3|\delta_{thl}|J_{3c})]-\xi_l[|\delta_{thu}|^2J_{11}+a_3^2J_{1l}+2(|\delta_{thu}||\delta_{thl}|J_{12}$$
$$-a_3|\delta_{thu}|J_{3c}-a_3|\delta_{thl}|J_{2l})]+\frac{1}{h}\Bigg[t(|\delta_{wu}|+|\delta_{wl}|)$$
$$\cdot\left(|\delta_{thu}|^2+|\delta_{thl}|^2+\frac{8\pi^2a_3^2}{3}\right)$$
$$+2(|\delta_{thu}||\delta_{thl}|J_{13}-a_3|\delta_{thu}|J_{3u}-a_3|\delta_{thl}|J_{3l})$$
$$-\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}(|\delta_{thl}|^2J_{24}+a_3^2J_{5u}+2[|\delta_{thu}||\delta_{thl}|J_{26}$$
$$-a_3|\delta_{thu}|J_{3d}-a_3|\delta_{thl}|J_{3e}])$$
$$-\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}(|\delta_{thu}|^2J_{25}+a_3^2J_{5l}+2[|\delta_{thu}||\delta_{thl}|J_{27}$$
$$-a_3|\delta_{thu}|J_{3f}-a_3|\delta_{thl}|J_{3g}])\Bigg]\Bigg\}$$

$$I_{3d}=3\Bigg\{\Big[\pi(\xi_u^2+\xi_l^2)+2\xi_u\xi_lJ_{13}\Big]\Big[1+\frac{t}{\pi h}(|\delta_{wu}|+|\delta_{wl}|)\Big]$$
$$+b_1(\xi_u^2J_{36}+\xi_l^2J_{37}+2\xi_u\xi_lJ_{38})+a_2[|\delta_{thu}|\xi_l(\xi_lJ_{12}+2\xi_uJ_{11})$$
$$+|\delta_{thl}|\xi_u(\xi_uJ_{11}+2\xi_lJ_{12})]$$
$$-\frac{1}{h}\Bigg[\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}\xi_l(\xi_lJ_{24}+2\xi_uJ_{26})$$
$$+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}\xi_u(\xi_uJ_{25}+2\xi_lJ_{27})\Bigg]\Bigg\}$$

$$I_{3e}=\frac{3}{h^2}\Bigg\{\frac{2t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)^2+\pi\Bigg[\left(\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}\right)^2$$
$$+\left(\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}\right)^2\Bigg]$$
$$+\frac{2|\delta_{wu}||\delta_{wl}|}{\Omega_u\Omega_l}\sin\left(\frac{\Omega_ut}{2}\right)\sin\left(\frac{\Omega_lt}{2}\right)(J_{21}+b_1J_{43}-a_3J_{3a})$$
$$+b_1\Bigg\{-\frac{2t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)\Bigg[\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}J_{6u}$$
$$+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}J_{6l}\Bigg]+\Bigg[\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}\Bigg]^2J_{41}$$
$$+\Bigg[\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}\Bigg]^2J_{42}\Bigg\}-a_2\Bigg\{\frac{2t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)$$
$$\cdot\Bigg[|\delta_{thu}|\left(\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}J_{14}+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}J_{16}\right)$$
$$+|\delta_{thl}|\left(\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}J_{15}+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}J_{17}\right)\Bigg]$$
$$-a_3\Bigg[2t^2(|\delta_{wu}|+|\delta_{wl}|)^2-\frac{2t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)$$
$$\cdot\left(\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}J_{7u}+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}J_{7l}\right)$$
$$+\left(\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}\right)^2J_{8u}+\left(\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}\right)^2J_{8l}\Bigg]\Bigg\}$$
$$+\frac{2t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)\Bigg[\frac{|\delta_{wu}|\sin\left(\frac{\Omega_ut}{2}\right)}{\Omega_u}(\xi_uJ_{14}+\xi_lJ_{15})$$
$$+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}(\xi_uJ_{16}+\xi_lJ_{17})\Bigg]$$
$$-\Bigg[\frac{|\delta_{wl}|\sin\left(\frac{\Omega_lt}{2}\right)}{\Omega_l}\Bigg]^2J_{28}(a_2|\delta_{thu}|+\xi_u)$$

$$-\left[\frac{|\delta_{wu}|\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u}\right]^2 J_{29}(a_2(|\delta_{thl}|+\xi_l)\Bigg\}$$

$$I_{6a}=6\Big\{a_2b_1(|\delta_{thu}|J_{9u}+|\delta_{thl}|J_{9l}-a_3J_{3h})-a_3[\xi_u(\pi|\delta_{thu}|$$
$$+|\delta_{thl}|J_{13}-a_3J_{3u})+\xi_l(|\delta_{thu}|J_{13}+\pi|\delta_{thl}|-a_3J_{3l})]$$
$$+\frac{1}{h}\left[\frac{|\delta_{wu}|\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u}(\xi_uJ_{14}+\xi_lJ_{15})\right.$$
$$\left.+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_l t}{2}\right)}{\Omega_l}(\xi_uJ_{16}+\xi_lJ_{17})\right]$$
$$-\frac{a_2}{h}\left[\frac{|\delta_{wu}|\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u}(|\delta_{thu}|J_{14}+|\delta_{thl}|J_{15}-a_3J_{7u})\right.$$
$$+\frac{|\delta_{wl}|\sin\left(\frac{\Omega_l t}{2}\right)}{\Omega_l}(|\delta_{thu}|J_{16}+|\delta_{thl}|J_{17}-a_3J_{7l})$$
$$\left.+2\pi a_3t(|\delta_{wu}|+|\delta_{wl}|)\right]\Big\}$$

$$I_{6b}=-6b_1\Big\{a_2[\xi_u(|\delta_{thu}|J_{36}+|\delta_{thl}|J_{38}-a_3J_{4u})+\xi_l(|\delta_{thu}|J_{38}$$
$$+|\delta_{thl}|J_{37}-a_3J_{4l})]+\frac{1}{h}\left[\frac{t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)(\xi_uJ_{9u}+\xi_lJ_{9l})\right.$$
$$-\frac{|\delta_{wu}|\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u}(\xi_uJ_{44}+\xi_lJ_{46}-J_{39})$$
$$\left.-\frac{|\delta_{wl}|\sin\left(\frac{\Omega_l t}{2}\right)}{\Omega_l}(\xi_uJ_{45}+\xi_lJ_{47}-J_{40})\right]\Big\}$$

$$I_{6c}=\frac{6a_2}{h}\Big\{b_1\left[\frac{t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)(|\delta_{thu}|J_{9u}+|\delta_{thl}|J_{9l}-a_3J_{3h})\right.$$
$$-\frac{|\delta_{wu}|\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u}(|\delta_{thu}|J_{44}+|\delta_{thl}|J_{46}-a_3J_{48})$$
$$\left.-\frac{|\delta_{wl}|\sin\left(\frac{\Omega_l t}{2}\right)}{\Omega_l}|\delta_{thu}|J_{45}+|\delta_{thl}|J_{47}-a_3J_{49})\right]$$
$$-\xi_u\left[\frac{t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)(\pi|\delta_{thu}|+|\delta_{thl}|J_{13}-a_3J_{3u})\right.$$
$$-\frac{|\delta_{wu}|\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u}(|\delta_{thl}|J_{26}-a_3J_{3d})$$
$$\left.-\frac{|\delta_{wl}|\sin\left(\frac{\Omega_l t}{2}\right)}{\Omega_l}(|\delta_{thu}|J_{25}+|\delta_{thl}|J_{27}-a_3J_{3f})\right]$$
$$-\xi_l\left[\frac{t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)(|\delta_{thu}|J_{13}+\pi|\delta_{thl}|-a_3J_{3l})\right.$$
$$-\frac{|\delta_{wu}|\sin\left(\frac{\Omega_u t}{2}\right)}{\Omega_u}(|\delta_{thu}|J_{26}+|\delta_{thl}|J_{24}-a_3J_{3e})$$
$$\left.-\frac{|\delta_{wl}|\sin\left(\frac{\Omega_l t}{2}\right)}{\Omega_l}(|\delta_{thu}|J_{27}-a_3J_{3g})\right]\Big\}$$

$$I_{6d}=-6b_1(\xi_uJ_{9u}+\xi_lJ_{9l})$$

where

$$J_{14}=\pi\cos\left(\frac{\Omega_u t}{2}+\Phi_u\right)$$

$$J_{15}=\delta_{(n_u)(n_l)}\pi\cos\left[\left(\Omega_l-\frac{\Omega_u}{2}\right)t+\Phi_u\right]$$

$$J_{16}=\delta_{(n_u)(n_l)}\pi\cos\left[\left(\Omega_u-\frac{\Omega_l}{2}\right)t-\Phi_l\right]$$

$$J_{17}=\pi\cos\left(\frac{\Omega_l t}{2}-\Phi_l\right)$$

$$J_{21}=\delta_{(n_u)(n_l)}\pi\cos\left[(\Omega_u-\Omega_l)\frac{t}{2}-\Phi_u-\Phi_l\right]$$

$$J_{22}=\delta_{(2n_u)(n_l)}\frac{\pi}{2}\sin\left[\left(\frac{\Omega_l}{2}-\Omega_u\right)t+\phi+2\Phi_u+\Phi_l\right]$$

$$J_{23}=\delta_{(2n_l)(n_u)}\frac{\pi}{2}\sin\left[\left(\frac{\Omega_u}{2}-\Omega_l\right)t+\phi-2\Phi_l-\Phi_u\right]$$

$$J_{24}=\delta_{(2n_l)(n_u)}\frac{\pi}{2}\sin\left[\left(\frac{\Omega_u}{2}-2\Omega_l\right)t+\phi-\Phi_u\right]$$

$$J_{25}=\delta_{(2n_u)(n_l)}\frac{\pi}{2}\sin\left[\left(\frac{\Omega_l}{2}-2\Omega_u\right)t+\phi+\Phi_l\right]$$

$$J_{26}=\delta_{(2n_u)(n_l)}\frac{\pi}{2}\sin\left[\left(\Omega_l-\frac{3\Omega_u}{2}\right)t+\phi+\Phi_u\right]$$

$$J_{27}=\delta_{(2n_l)(n_u)}\frac{\pi}{2}\sin\left[\left(\Omega_u-\frac{3\Omega_l}{2}\right)t+\phi-\Phi_l\right]$$

$$J_{28}=\delta_{(2n_l)(n_u)}\frac{\pi}{2}\sin\left[(\Omega_u-\Omega_l)t+\phi-2\Phi_l\right]$$

$$J_{29}=\delta_{(2n_u)(n_l)}\frac{\pi}{2}\sin\left[(\Omega_l-\Omega_u)t+\phi+2\Phi_u\right]$$

$$J_{31}=-\delta_{(2)(n_u)}\frac{\pi}{2}\sin(\Omega_u t-\phi-2\overline{\Psi})$$

$$J_{32}=-\delta_{(2)(n_l)}\frac{\pi}{2}\sin(\Omega_l t-\phi-2\overline{\Psi})$$

$$J_{33}=\frac{\pi}{2}[2\pi-\sin(2\overline{\Psi})]$$

$$J_{34}=-\delta_{(2)(n_u)}\frac{\pi}{2}\sin\left(\frac{\Omega_u t}{2}-\phi-\Phi_u-2\overline{\Psi}\right)$$

$$J_{35}=-\delta_{(2)(n_l)}\frac{\pi}{2}\sin\left(\frac{\Omega_l t}{2}-\phi+\Phi_l-2\overline{\Psi}\right)$$

$$J_{39}=\delta_{(n_u)(1)}\pi\sin\left(-\frac{\Omega_u t}{2}+\overline{\Psi}+\phi+\Phi_u\right)$$

$$J_{40}=\delta_{(n_l)(1)}\pi\sin\left(-\frac{\Omega_l t}{2}+\overline{\Psi}+\phi-\Phi_l\right)$$

$$J_{41}=-\delta_{(2n_u)(1)}\frac{\pi}{2}\cos(\Omega_u t-2\phi-2\Phi_u-\overline{\Psi})$$

$$J_{42}=-\delta_{(2n_l)(1)}\frac{\pi}{2}\cos(\Omega_l t-2\phi+2\Phi_l-\overline{\Psi})$$

$$J_{43}=\frac{\pi}{2}\Big\{\delta_{(n_u+1)(n_l)}\cos\left[(\Omega_u-\Omega_l)\frac{t}{2}+\overline{\Psi}-\Phi_u-\Phi_l\right]$$
$$+\delta_{(n_l+1)(n_u)}\cos\left[(\Omega_u-\Omega_l)\frac{t}{2}-\overline{\Psi}-\Phi_u-\Phi_l\right]\Big\}$$

$$J_{44}=-\delta_{(2n_u)(1)}\frac{\pi}{2}\cos\left(\frac{3\Omega_u t}{2}-2\phi-\Phi_u-\overline{\Psi}\right)$$

$$J_{45}=\frac{\pi}{2}\Big\{\delta_{(n_u+1)(n_l)}\cos\left[\left(\Omega_u-\frac{\Omega_l}{2}\right)t+\overline{\Psi}-\Phi_l\right]$$
$$+\delta_{(n_l+1)(n_u)}\cos\left[\left(\Omega_u-\frac{\Omega_l}{2}\right)t-\overline{\Psi}-\Phi_l\right]\Big\}$$

$$J_{46}=\frac{\pi}{2}\Big\{\delta_{(n_u+1)(n_l)}\cos\left[\left(\frac{\Omega_u}{2}-\Omega_l\right)t+\overline{\Psi}-\Phi_u\right]$$
$$+\delta_{(n_l+1)(n_u)}\cos\left[\left(\frac{\Omega_u}{2}-\Omega_l\right)t-\overline{\Psi}-\Phi_u\right]\Big\}$$

$$J_{47}=-\delta_{(2n_l)(1)}\frac{\pi}{2}\cos\left(\frac{3\Omega_l t}{2}-2\phi+\Phi_l-\overline{\Psi}\right)$$

$$J_{48}=\begin{cases}-\frac{\pi}{2}\left[\cos\left(\frac{\Omega_u t}{2}-\phi-\Phi_u+\overline{\Psi}\right)+2\pi\sin\left(\frac{\Omega_u t}{2}-\phi-\Phi_u-\overline{\Psi}\right)\right] & \text{when } n_u=1\\ -\pi\left[\frac{1}{n_u+1}\cos\left(\frac{\Omega_u t}{2}-\phi-\Phi_u+\overline{\Psi}\right)+\frac{1}{n_l-1}\cos\left(\frac{\Omega_u t}{2}-\phi-\Phi_u-\overline{\Psi}\right)\right] & \text{when } n_u\neq 1\end{cases}$$

$$J_{49}=\begin{cases} -\frac{\pi}{2}\left[\cos\left(\frac{\Omega_l t}{2}-\phi+\Phi_l+\bar{\Psi}\right)+2\pi\sin\left(\frac{\Omega_l t}{2}-\phi+\Phi_l-\bar{\Psi}\right)\right] & \text{when } n_l=1 \\ -\pi\left[\frac{1}{n_l+1}\cos\left(\frac{\Omega_l t}{2}-\phi+\Phi_l+\bar{\Psi}\right)+\frac{1}{n_l-1}\cos\left(\frac{\Omega_l t}{2}-\phi+\Phi_l-\bar{\Psi}\right)\right] & \text{when } n_l\neq 1 \end{cases}$$

$$J_{1u}=-\frac{4\pi}{n_u}\left[\pi\cos(\Omega_u t-\phi)+\frac{1}{n_u}\sin(\Omega_u t-\phi)\right]$$

$$J_{1l}=-\frac{4\pi}{n_l}\left[\pi\cos(\Omega_l t-\phi)+\frac{1}{n_l}\sin(\Omega_l t-\phi)\right]$$

$$J_{2u}=\frac{\pi}{2}\left[2\pi+\frac{1}{n_u}\sin(2\Omega_u t-2\phi)\right]$$

$$J_{2l}=\frac{\pi}{2}\left[2\pi+\frac{1}{n_l}\sin(2\Omega_l t-2\phi)\right]$$

$$J_{3u}=-\frac{2\pi}{n_u}\cos(\Omega_u t-\phi)$$

$$J_{3l}=-\frac{2\pi}{n_l}\cos(\Omega_l t-\phi)$$

$$J_{4u}=\begin{cases} -\frac{\pi}{2}\left[\cos(\Omega_u t-\phi+\bar{\Psi})+2\pi\sin(\Omega_u t-\phi-\bar{\Psi})\right] & \text{when } n_u=1 \\ -\pi\left[\frac{1}{n_u+1}\cos(\Omega_u t-\phi+\bar{\Psi})+\frac{1}{n_u-1}\cos(\Omega_u t-\phi-\bar{\Psi})\right] & \text{when } n_u\neq 1 \end{cases}$$

$$J_{4l}=\begin{cases} -\frac{\pi}{2}\left[\cos(\Omega_l t-\phi+\bar{\Psi})+2\pi\sin(\Omega_l t-\phi-\bar{\Psi})\right] & \text{when } n_l=1 \\ -\pi\left[\frac{1}{n_l+1}\cos(\Omega_l t-\phi+\bar{\Psi})+\frac{1}{n_l-1}\cos(\Omega_l t-\phi-\bar{\Psi})\right] & \text{when } n_l\neq 1 \end{cases}$$

$$J_{5u}=-\frac{4\pi}{n_u}\left[\pi\cos\left(\frac{\Omega_u t}{2}-\phi-\Phi_u\right)+\frac{1}{n_u}\sin\left(\frac{\Omega_u t}{2}-\phi-\Phi_u\right)\right]$$

$$J_{5l}=-\frac{4\pi}{n_l}\left[\pi\cos\left(\frac{\Omega_l t}{2}-\phi+\Phi_l\right)+\frac{1}{n_l}\sin\left(\frac{\Omega_l t}{2}-\phi+\Phi_l\right)\right]$$

$$J_{6u}=\delta_{(n_u)(1)}\pi\sin\left(-\frac{\Omega_u t}{2}+\phi+\Phi_u+\bar{\Psi}\right)$$

$$J_{6l}=\delta_{(n_l)(1)}\pi\sin\left(-\frac{\Omega_l t}{2}+\phi-\Phi_l+\bar{\Psi}\right)$$

$$J_{7u}=-\frac{2\pi}{n_u}\cos\left(\frac{\Omega_u t}{2}-\phi-\Phi_u\right)$$

$$J_{7l}=-\frac{2\pi}{n_l}\cos\left(\frac{\Omega_l t}{2}-\phi+\Phi_l\right)$$

$$J_{8u}=\frac{\pi}{2}\left[2\pi+\frac{1}{n_u}\sin(\Omega_u t-2\phi-2\Phi_u)\right]$$

$$J_{8l}=\frac{\pi}{2}\left[2\pi+\frac{1}{n_l}\sin(\Omega_l t-2\phi+2\Phi_l)\right]$$

$$J_{3a}=\begin{cases} \frac{\pi}{2}\left\{2\pi\cos[(\Omega_l-\Omega_u)t+\Phi_u+\Phi_l]+\frac{1}{n_u}\sin[(\Omega_u+\Omega_l)t-2\phi-\Phi_u+\Phi_l]\right\} & \text{when } n_u=n_l \\ \pi\left\{\frac{1}{n_u-n_l}\sin[(\Omega_l-\Omega_u)t+\Phi_u+\Phi_l]+\frac{1}{n_u+n_l}\sin[(\Omega_u+\Omega_l)t-2\phi-\Phi_u+\Phi_l]\right\} & \text{when } n_u\neq n_l \end{cases}$$

$$J_{3b}=4\pi(\cos\bar{\Psi}-\pi\sin\bar{\Psi})$$

$$J_{3c}=\begin{cases} \frac{\pi}{2}\left\{2\pi\cos[(\Omega_l-\Omega_u)t]+\frac{1}{n_u}\sin[(\Omega_u+\Omega_l)t-2\phi]\right\} & \text{when } n_u=n_l \\ \pi\left\{\frac{1}{n_u-n_l}\sin[(\Omega_l-\Omega_u)t]+\frac{1}{n_u+n_l}\sin[(\Omega_u+\Omega_l)t-2\phi]\right\} & \text{when } n_u\neq n_l \end{cases}$$

$$J_{3d}=\frac{\pi}{2}\left[2\pi\cos\left(\frac{\Omega_u t}{2}+\Phi_u\right)+\frac{1}{n_u}\sin\left(\frac{3\Omega_u t}{2}-2\phi-\Phi_u\right)\right]$$

$$J_{3e}=\begin{cases} \frac{\pi}{2}\left\{2\pi\cos\left[\left(\Omega_l-\frac{\Omega_u}{2}\right)t+\Phi_u\right]+\frac{1}{n_u}\sin\left[\left(\frac{\Omega_u}{2}+\Omega_l\right)t-2\phi-\Phi_u\right]\right\} & \text{when } n_u=n_l \\ \pi\left\{\frac{1}{n_u-n_l}\sin\left[\left(\Omega_l-\frac{\Omega_u}{2}\right)t+\Phi_u\right]+\frac{1}{n_u+n_l}\sin\left[\left(\frac{\Omega_u}{2}+\Omega_l\right)t-2\phi-\Phi_u\right]\right\} & \text{when } n_u\neq n_l \end{cases}$$

$$J_{3f}=\begin{cases} \frac{\pi}{2}\left\{2\pi\cos\left[\left(\Omega_u-\frac{\Omega_l}{2}\right)t-\Phi_l\right]+\frac{1}{n_l}\sin\left[\left(\frac{\Omega_l}{2}+\Omega_u\right)t-2\phi+\Phi_l\right]\right\} & \text{when } n_u=n_l \\ \pi\left\{\frac{1}{n_u-n_l}\sin\left[\left(\Omega_u-\frac{\Omega_l}{2}\right)t-\Phi_l\right]+\frac{1}{n_u+n_l}\sin\left[\left(\frac{\Omega_l}{2}+\Omega_u\right)t-2\phi+\Phi_l\right]\right\} & \text{when } n_u\neq n_l \end{cases}$$

$$J_{3g}=\frac{\pi}{2}\left[2\pi\cos\left(\frac{\Omega_l t}{2}-\Phi_l\right)+\frac{1}{n_l}\sin\left(\frac{3\Omega_l t}{2}-2\phi+\Phi_l\right)\right]$$

$$J_{3h}=-2\pi\sin\bar{\Psi}$$

$$b_1=a_1|\bar{\varepsilon}|$$

where the symbol, δ is defined as follows:

$$\delta_{(i)(j)}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$$